Weak Values with Decoherence

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We introduce a weak operator associated with weak values and give a general framework of quantum operations to the weak operator in parallel with the Kraus representation of the completely positive map for the density operator. The decoherence effect is also investigated in terms of the weak measurement by a shift of a probe wave function of a continuous variable.

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I. INTRODUCTION AND CONCLUSION

What is an observable? The observable is conventionally defined as the self-adjoint operator [1]. However, we can measure the non self-adjoint operators like the momentum operator on a half line and the time operator in some senses [2]. Can we extend the quantum measurement theory? The possible answer may be that the measurement outcome changes to weak values defined as follows. For an operator A, the weak value $\langle A \rangle_w$ is defined as

$$\langle A \rangle_w = \frac{\langle f | U(t_f, t) A U(t, t_i) | i \rangle}{\langle f | U(t_f, t_i) | i \rangle} \in \mathbb{C}, \tag{1}$$

where $|i\rangle$ and $\langle f|$ are normalized pre-selected ket and post-selected bra state vectors, respectively. Here, $U(t_2, t_1)$ is an evolution operator from the time t_1 to t_2. The weak value of an observable is experimentally accessible by weak measurements shown in the details in Sec. III as theoretically analyzed by Aharonov and his collaborators [3–5] and recently experimentally demonstrated (e. g., see [6, 7]).

Our aim is to extend the weak value theory to consider decoherence. Here, the result was to explicitly describe the quantum operation $\mathcal{E}(W) = \sum_m E_m W F_m^\dagger$ for the weak operator W to formally describe the weak value [8]. The weak operator is a useful tool to compactly describe the effect of decoherence to the weak values just as the density operator is to the expectation value in the standard theory of decoherence. Furthermore, the amount of the effect due to the environment in the weak measurement is exactly given by the weak value defined by the quantum operation of the weak operator $\mathcal{E}(W)$.

II. QUANTUM OPERATIONS FOR WEAK OPERATORS

Let us now define a weak operator [8] as

$$W(t) := |\psi(t)\rangle\langle\phi(t)|, \tag{2}$$

based on the two-state vector formalism by Aharonov and Vaidman [4] and define

$$\langle A \rangle_W := \frac{\mathrm{Tr}(AW)}{\mathrm{Tr}(W)}, \tag{3}$$

for an observable A corresponding to the weak value of the observable A [3] as the above. The weak value is an analog of a probability distribution, and so is the weak operator that of the density operators. We discuss a state change in terms of the weak operator and define a map X as

$$X(|\alpha\rangle, |\beta\rangle) := (\mathcal{E} \otimes I)\left(|\alpha\rangle\langle\beta|\right), \tag{4}$$

for an arbitrary $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_s \otimes \mathcal{H}_e$. Then, we obtain the following theorem on the change of the weak operator *a la* one of the density operator.

Theorem 1. *For any weak operator* $W = |\psi(t)\rangle_s\langle\phi(t)|$*, we expand*

$$|\psi(t)\rangle_s = \sum_m \psi_m |\alpha_m\rangle_s,$$
$$|\phi(t)\rangle_s = \sum_m \phi_m |\beta_m\rangle_s, \tag{5}$$

with fixed complete orthonormal sets $\{|\alpha_m\rangle_s\}$ *and* $\{|\beta_m\rangle_s\}$*. Then, a change of the weak operator can be written as*

$$\mathcal{E}\left(|\psi(t)\rangle_s\langle\phi(t)|\right) = {}_e\langle\tilde{\psi}(t)| X(|\alpha\rangle, |\beta\rangle) |\tilde{\phi}(t)\rangle_e, \tag{6}$$

where

$$|\tilde{\psi}(t)\rangle_e = \sum_k \psi_k^* |\alpha_k\rangle_e,$$
$$|\tilde{\phi}(t)\rangle_e = \sum_k \phi_k^* |\beta_k\rangle_e, \tag{7}$$

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and $|\alpha\rangle$ and $|\beta\rangle$ are maximally entangled states defined by $|\alpha\rangle := \sum_m |\alpha_m\rangle_s|\alpha_m\rangle_e$, $|\beta\rangle := \sum_m |\beta_m\rangle_s|\beta_m\rangle_e$. Here, $\{|\alpha_m\rangle_e\}$ and $\{|\beta_m\rangle_e\}$ are complete orthonormal sets corresponding to $\{|\alpha_m\rangle_s\}$ and $\{|\beta_m\rangle_s\}$, respectively.

We take the polar decomposition of the map X to obtain

$$X = \sigma u. \tag{8}$$

The unitary operator u is well-defined on $\mathcal{H}_s \otimes \mathcal{H}_e$ and σ is positive. From $\sigma = \sum |s_m\rangle\langle s_m|$, we can rewrite X as

$$\begin{aligned} X &= \sum_m |s_m\rangle\langle s_m|u \\ &= \sum_m |s_m\rangle\langle t_m|, \end{aligned} \tag{9}$$

where

$$\langle t_m| = \langle s_m|u. \tag{10}$$

Similar to the Kraus operator, we define the two operators, E_m and $F_m^\dagger$, as

$$E_m|\psi(t)\rangle_s := {}_e\langle\tilde{\psi}(t)|s_m\rangle, \tag{11}$$

$${}_s\langle\phi(t)|F_m^\dagger := \langle t_m|\tilde{\phi}(t)\rangle_e, \tag{12}$$

where $|\tilde{\psi}(t)\rangle_e$ and $|\tilde{\phi}(t)\rangle_e$ are defined in Eq. (7). Therefore, we obtain the Kraus form for the weak operator as

$$\sum_m E_m|\psi(t)\rangle_s\langle\phi(t)|F_m^\dagger = \mathcal{E}\left(|\psi(t)\rangle_s\langle\phi(t)|\right), \tag{13}$$

using Theorem 1. By linearity, we conclude

$$\mathcal{E}(W) = \sum_m E_m W F_m^\dagger. \tag{14}$$

Note that, in general, $\mathcal{E}(W)\mathcal{E}(W^\dagger) \neq \mathcal{E}(\rho)$ although $\rho = WW^\dagger$. Furthermore, Eq. (14) can be derived using the quantum comb [9].

It is well established that the trace preservation, $\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}\,\rho = 1$ for all ρ, implies that $\sum_m E_m^\dagger E_m = 1$. This argument for the density operator $\rho = WW^\dagger$ applies also for $W^\dagger W$ to obtain $\sum_m F_m^\dagger F_m = 1$. Therefore, we can express the Kraus operators,

$$\begin{aligned} E_m &= {}_e\langle e_m|U|e_i\rangle_e, \\ F_m^\dagger &= {}_e\langle e_f|V|e_m\rangle_e, \end{aligned} \tag{15}$$

where $U = U(t, t_i)$ and $V = U(t_f, t)$ are the evolution operators, which act on $\mathcal{H}_s \otimes \mathcal{H}_e$. $|e_i\rangle$ and $|e_f\rangle$ are some basis vectors and $|e_m\rangle$ is a complete set of basis vectors with $\sum_m |e_m\rangle\langle e_m| = 1$. We can compute

$$\begin{aligned} \sum_m F_m^\dagger E_m &= \sum_m {}_e\langle e_f|V|e_m\rangle_e\langle e_m|U|e_i\rangle_e \\ &= {}_e\langle e_f|VU|e_i\rangle_e. \end{aligned} \tag{16}$$

The above equality (16) may be interpreted as a decomposition of the history analogous to the decomposition of unity because

$${}_e\langle e_f|VU|e_i\rangle_e = {}_e\langle e_f|S|e_i\rangle_e = S_{fi} \tag{17}$$

is the S-matrix element. The meaning of the basis $|e_i\rangle$ and $|e_f\rangle$ will be clear in Sec. IV.

III. WEAK MEASUREMENT—REVIEW

In this section, we recapitulate the idea of the weak measurement [3, 4, 10]. Consider a target system and a probe defined in the Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_p$. The interaction of the target system and the probe is assumed to be weak and instantaneous,

$$H_{int}(t) = g\delta(t - t_0)(A \otimes P), \tag{18}$$

where an observable A is defined in $\mathcal{H}_s$, while P is the momentum operator of the probe. The time evolution operator becomes

$$e^{-ig(A\otimes P)}. \tag{19}$$

Suppose the probe state is initially $\xi(q) \in \mathbb{R}$ in the coordinate representation with the probe position q. For the transition from the pre-selected state $|i\rangle$ to the post-selected state $|f\rangle$, the probe wave function becomes

$$\langle f|Ve^{-ig(A\otimes P)}U|i\rangle\xi(q), \tag{20}$$

which is in the weak coupling case [11],

$$\begin{aligned} &\langle f|V[1 - ig(A \otimes P)]U|i\rangle\xi(q) \\ &\approx \langle f|VU|i\rangle\xi(q) - g\langle f|VAU|i\rangle\xi'(q) \\ &\approx \langle f|VU|i\rangle\xi\left(q - g\langle A\rangle_w\right). \end{aligned} \tag{21}$$

So that the shift of the expectation value is the real part of the weak value, $g \cdot \mathrm{Re}[\langle A\rangle_w]$. The shift of the momentum distribution can be similarly calculated to give $2g \cdot Var(p) \cdot \mathrm{Im}[\langle A\rangle_w]$, where $Var(p)$ is the variance of the probe momentum before the interaction.

The main concept of weak measurement is a time symmetric description on quantum measurement and not a destruction of the quantum state [5, 12]. First, according to the Copenhagen doctrine, quantum measurement must be described time-asymmetrically. However, by the post-selection, we can describe quantum measurement time-symmetrically such as the well-known physical fundamental equations, e.g., the Newton equation and the Schrödinger equation. Second, we only get minicule information on the quantum state in the target system by weak measurement at once time to extract its information coupled to the probe. When we obtain the weak values by weak measurement, we have to repeat this measurement procedure instead of not destroying the quantum system in the target system. Roughly speaking, weak measurement seems to peep at the quantum state on the way to the time evolution as if we undertake *in-vivo* experiment of the quantum system [13]. Furthermore, from the obtained experimental quantity, which is the weak value, we can evaluate the quantum state of the future and the past as if we compile our history and guess the future event from the past experiences.

IV. WEAK MEASUREMENT WITH DECOHERENCE

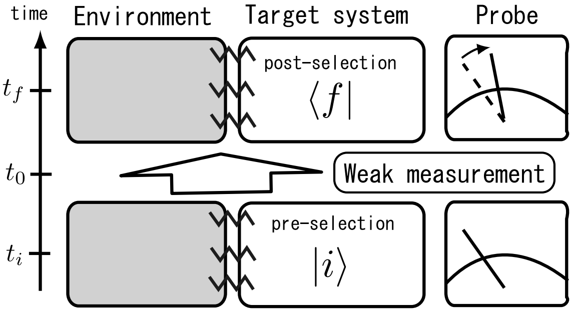

FIG. 1: A weak measurement model with the environment. The environment affects the target system as a noise but does not affect the probe. The weak measurement for the target system and the probe brings about the shift of the probe position at t_0.

Let us consider a target system coupled with an environment and a general weak measurement for the compound of the target system and the environment. We assume that there is no interaction between the probe and the environment. This situation is illustrated in Fig. 1. The Hamiltonian for the target system and the environment is given by

$$H = H_0 \otimes I_e + H_1, \tag{22}$$

where H_0 acts on the target system $\mathcal{H}_s$ and the identity operator I_e is for the environment $\mathcal{H}_e$, while H_1 acts on $\mathcal{H}_s \otimes \mathcal{H}_e$. The evolution operators $U = U(t, t_i)$ and $V = U(t_f, t)$ can be expressed by $U = U_0 K(t_0, t_i)$ and $V = K(t_f, t_0)V_0$ where U_0 and V_0 are the evolution operators forward in time and backward in time, respectively, by the target Hamiltonian H_0. K's are the evolution operators in the interaction picture,

$$\begin{aligned} K(t_0, t_i) &= \mathcal{T} e^{-i\int_{t_i}^{t_0} dt U_0^\dagger H_1 U_0}, \\ K(t_f, t_0) &= \overline{\mathcal{T}} e^{-i\int_{t_0}^{t_f} dt V_0 H_1 V_0^\dagger}, \end{aligned} \tag{23}$$

where $\mathcal{T}$ and $\overline{\mathcal{T}}$ stand for the time-ordering and anti time-ordering products.

Let the initial and final environmental states be $|e_i\rangle$ and $|e_f\rangle$, respectively. The probe state becomes

$$N\xi\left(q - g\frac{\langle f|\langle e_f|K(t_f,t_0)V_0AU_0K(t_0,t_i)|e_i\rangle|i\rangle}{N}\right), \tag{24}$$

where $N = \langle f|\langle e_f|K(t_f,t_0)V_0U_0K(t_0,t_i)|e_i\rangle|i\rangle$ is the normalization factor. We define the dual quantum operation as

$$\begin{aligned} \mathcal{E}^*(A) &:= \langle e_f|K(t_f,t_0)V_0AU_0K(t_0,t_i)|e_i\rangle \\ &= \sum_m V_0F_m^\dagger AE_mU_0, \end{aligned} \tag{25}$$

where $F_m^\dagger := V_0^\dagger\langle e_f|K(t_f,t_0)|e_m\rangle V_0$ and $E_m := U_0\langle e_m|K(t_0,t_i)|e_i\rangle U_0^\dagger$ are the Kraus operators introduced in the previous section (15). Here, we have inserted the completeness relation $\sum_m |e_m\rangle\langle e_m| = 1$ with $|e_m\rangle$ being not necessarily orthogonal. The basis $|e_i\rangle$ and $|e_f\rangle$ are the initial and final environmental states, respectively. This provides the meaning of $|e_i\rangle$ and $|e_f\rangle$ as alluded to Sec. II. Thus, we obtain the wave function of the probe as

$$\xi\left(q - g\frac{\langle f|\mathcal{E}^*(A)|i\rangle}{N}\right) = \xi(q - g\langle A\rangle_{\mathcal{E}(W)}), \tag{26}$$

with $N = \langle f|\mathcal{E}^*(I)|i\rangle$ up to the overall normalization factor. This is the main result of this subsection. The shift of the expectation value of the position operator on the probe is

$$\delta q = g \cdot \mathrm{Re}[\langle A\rangle_{\mathcal{E}(W)}]. \tag{27}$$

From an analogous discussion, we obtain the shift of the expectation value of the momentum operator on the probe as $\delta p = 2g \cdot Var(p) \cdot \mathrm{Im}[\langle A\rangle_{\mathcal{E}(W)}]$. Thus, we have shown that the probe shift in the weak measurement is exactly given by the weak value defined by the quantum operation of the weak operator due to the environment.

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[13] On the other hand, the projective measurement, or strong measurement, seems to be *in-vitro* experiment.